DEVELOPING THE CALCULUS

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Abstract: *Beginning as early as 400 B.C., mathematicians used the concept of limit, which is essential to the theory of the branch of mathematics that is today known as Calculus, to aid them in determining areas. While Sir Isaac Newton and Gottfried Wilhelm Leibniz are credited as being the co-inventors of the subject, it is necessary to view the development of Calculus as the work of many mathematicians, including Joseph Louis Lagrange, Augustin-Louis Cauchy, Bernhard Riemann, and many others. Newton and Leibniz were the first to axiomatically define the subject; others would improve upon their work in order to build the subject of Calculus to be that which it is today.*

The subject of Calculus is made up of four main concepts: the limit, continuity, the derivative, and the integral. Perhaps the two most sought after concepts in the history of the subject were the integral and the derivative, as they were a necessity in finding out information needed about the physical world. For example, computation of the derivative could provide the instantaneous velocity of a moving particle; integration was useful in finding the area of curved regions.

Before any such computations could be made, methods with which to compute tangents and areas had to be found. Although Sir Isaac Newton and Gottfried Wilhelm Leibniz are credited with "inventing" Calculus and presenting, for the first time, ways to find tangent lines and areas regardless of the curve, the development of Calculus must be viewed as a long, tedious process that would last for centuries. After Newton and Leibniz laid the foundation for defining what limit, continuity, derivative, and integral might mean, it would take another century for mathematicians such as Augustin-Louis Cauchy, Bernhard Riemann, and others to provide the world with rigorous, more exact definitions.

One of the earliest examples of the use of limits was the "method of exhaustion." It was once impossible to determine the area of a circle, but the mathematician Eudoxus (408-355 B.C.) observed a pattern: As the number of sides of a regular inscribed polygon became larger, the figures looked more and more circular. Furthermore, the left over area between the polygon and the circle decreased as the number of sides increased. "In

Eudoxean terms, the polygons are 'exhausting' the circle from within" [2, p. 29]. Mathematically, let P_n be a regular polygon with n sides inscribed within a circle. Then, $\lim_{n\to\infty} A(P_n) = A(C)$, where $A(P_n)$ is the area of the polygon with n sides, and A(C) is the area of the circle. However, the Greeks avoided openly taking the limit, as well as trying to explain what a limit was [3, p. 16].

Another ancient example of the naïve determination of a limit comes from the Greek mathematician, Archimedes (around 400 B.C.). He had stated as an axiom that taking any quantity and dividing it by two repeatedly, one could eventually reach a quantity less than any given number [1, p. 175]. Then, in modern terms, Archimedes would have said that, given any positive number x and a given quantity ε, there exists a natural number N, such that $\frac{x}{2^N} < \varepsilon$.

Over a thousand years after Archimedes and Eudoxus experimented with their notion of limit, mathematicians began trying to find generalized ways of finding tangents to curves, as well as their included areas. Pierre de Fermat was among the first people to approach the tangent line problem methodically. Around the year 1630, he was able to provide a method for determining the tangent to any given polynomial [9, p.140]. For example,

> Consider the parabola $y = x^2$ at the point (x, x^2). Let x + e be another point on the x-axis, (x + e, k) be a point on the tangent line to (x, x^2), and let s be the subtangent to the curve at (x, x^2). Then, using the similarity of triangles from geometry, $\frac{x^2}{s} = \frac{k}{s+e}$.
>
> Choosing e to be small enough, we say $k \cong (x+e)^2$.
>
> $$\frac{x^2}{s} \cong \frac{(x+e)^2}{s+e} \Rightarrow x^2(s+e) \cong s(x+e)^2$$
>
> $$\Rightarrow x^2 s + x^2 e \cong s(x+e)^2$$
>
> Then, $\Rightarrow x^2 e \cong s(x+e)^2 - sx^2 = s\left[(x+e)^2 - x^2\right]$
>
> $$\Rightarrow s \cong \frac{x^2 e}{\left[(x+e)^2 - x^2\right]} = \frac{x^2 e}{x^2 + 2xe + e^2 - x^2} = \frac{ex^2}{e(2x+e)} = \frac{x^2}{2x+e}$$
>
> $$\Rightarrow \frac{x^2}{s} \cong 2x + e$$
>
> Note that $\frac{x^2}{s}$ is the slope of the tangent to the parabola at the point (x, x^2). Then, Fermat concluded that one could simply discard the e, thus concluding that the tangent was 2x [9, pp.140-141].

While Fermat had essentially used the modern notion of computing a derivative, he did not realize that the method he had used to find the tangent could be made universal. In fact, he gave no name and no notation to what he was doing, although what he had essentially computed $\lim_{e\to 0}\frac{f(x+e)-f(x)}{e}$ is exactly the modern definition of the derivative [3, pp. 189-90].

Fermat was not the only person to generate tangents for specific types of functions during the seventeenth century. René François de Sluse (1622-1685) produced an algorithm for finding the curve f(x, y) = 0, where f is a polynomial. It was published in 1673 in the *Royal Transactions* of the Royal Society [1, p. 374]. Consider the following example [8, pp. 474-475]:

> Begin with the equation $x^5+bx^4-2q^2y^3+x^2y^3-b^2=0$
> Remove the constant term: $x^5+bx^4-2q^2y^3+x^2y^3=0$
> Next, make sure all terms with x are on the left and all terms with y are on the right. There will be some terms with both x and y on both sides:
> $x^5+bx^4+x^2y^3=2q^2y^3-x^2y^3$
> Multiply each x term on the left by its exponent:
> $5x^5+4bx^4+2x^2y^3=2q^2y^3-x^2y^3$
> Multiply each y term on the right by its exponent:
> $5x^5+4bx^4+2x^2y^3=6q^2y^3-3x^2y^3$
> Reduce each power of x on the left by 1 and multiply by t:
> $5x^4t+4bx^3t+2txy^3=6q^2y^3-3x^2y^3$
> Solve for the subtangent t:
> $t=\frac{6q^2y^3-3x^2y^3}{5x^4+4bx^3+2xy^3}$
> Then, the tangent is given by: $\frac{y}{t}=\frac{y}{\frac{6q^2y^3-3x^2y^3}{5x^4+4bx^3+2xy^3}}=\frac{5x^4+4bx^3+2xy^3}{6q^2y^2-3x^2y^2}$
> In modern terms, we can see that $\frac{dy}{dx}=\frac{-f_x(x,y)}{f_y(x,y)}$

The derivative was not the only concept developed through case-by-case examples during the 1600's. For example, Fermat had found the formula for the area under the curve $y = x^a$ on the interval [0, B], where a > -1, around the year 1636.

> Choose θ <1 but close to one, and consider the rectangles formed by B, θB, θB^2, θB^3, … , θB^n, each having height of B^a, $\theta^a B^a$, $\theta^{2a}B^a$, $\theta^{3a}B^a$, …, $\theta^{na}B^a$.

Then, the area under the curve can be approximated as follows:
Area = A(1st rectangle) + A(2nd rectangle) + A(3rd rectangle) + … + A(nth rectangle)

$$= B(1-\theta)B^a + B(\theta-\theta^2)\theta^a B^a + B(\theta^2-\theta^3)\theta^{2a} B^a + \ldots + B(\theta^{n-1}-\theta^n)\theta^{n-1} B^a$$

$$= B^{a+1}(1-\theta)(1+\theta^{a+1}+\theta^{2a+2}+\ldots+\theta^{na+n})$$

$$= B^{a+1}\left(\frac{1-\theta}{1-\theta^{a+1}}\right)$$

Since θ is assumed to be very close to 1, let $\varepsilon = 1-\theta$ be very small.

$$1-\theta = \varepsilon$$

Then, $\theta^{a+1} = 1-(a+1)\varepsilon + \ldots$

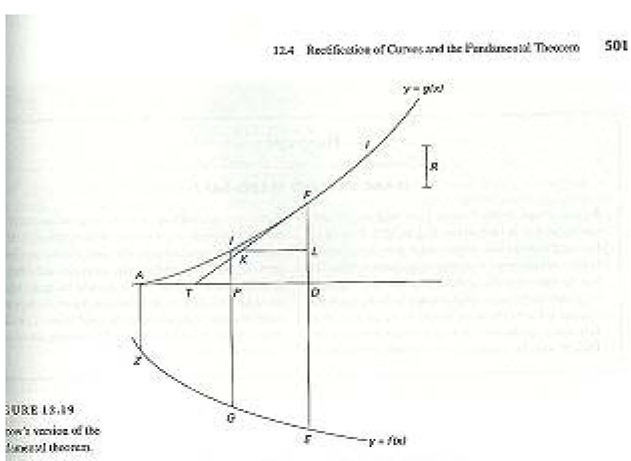
$$\Rightarrow \frac{1-\theta}{1-\theta^{a+1}} \approx \frac{\varepsilon}{(a+1)\varepsilon} = \frac{1}{a+1}$$

This sum approximates the area from below. Approximating the area from above will yield the same thing, and conclude that

Area = $\dfrac{B^{a+1}}{a+1}$, if a > -1 [7, pp. 32-33]

Another mathematician, Gilles Persone de Roberval, shared the discovery of this result with Fermat. In fact, both men were able to extend this conclusion to determine that the area under any curve $y = px^k$, where k is a given constant greater than 1, on the interval [0, y_o] was given by $A = \dfrac{x_o y_o}{k+1}$. Fermat most likely did not prove either result, as he was the mathematician most famous for not giving justification for his theories [8, pp. 481-482].

During this period of searching for tangents and areas corresponding to specific curves, Isaac Barrow, Newton's immediate predecessor, is said to have known the inverse relationship between derivatives and integrals. That is, he had knowledge of what is now known as the fundamental theorem of Calculus as early as 1669 when he published his most important work, *Lectiones opticae et geometricae* [4, pp. 301-302]. Consider the following diagram: [8, p. 501]



Begin with the curve ZGE, labeled for our purposes to be y = f(x)
Construct the curve AIF, labeled g(x), such that Rg(x) is equal to the area by f(x) between some fixed point, a, and the variable x.

Modern notation will show that $Rg(x) = \int_a^x f(x)$

Next, Barrow proved that the length t(x) of the subtangent to g(x) is

$$t(x) = \frac{Rg(x)}{f(x)}, \text{ or } g'(x) = \frac{g(x)}{t(x)} = \frac{g(x)}{\frac{Rg(x)}{f(x)}} = \frac{f(x)}{R}$$

Given these statements, we see that $\frac{d}{dx}\int_a^x f(x)dx = f(x)$ [8, pp. 500-501].

Observe that, using purely geometric methods, Barrow was able to arrive at this conclusion. He did not actually use the functional notation, nor did he realize the importance of the calculations that he was making. Therefore, it is not proper to say that Barrow invented Calculus. He merely stumbled upon the fundamental theorem by utilizing his knowledge of geometry [8, p. 503].

By the time Sir Isaac Newton and Gottfried Wilhelm Leibniz made their respective definitions of the Calculus concepts, "…practically all of the prominent mathematicians of Europe…could solve many of the problems in which elementary calculus is now used" [9, p. 139]. The subject could be traced as far back as 400 B.C., but nobody had ever clearly explored or developed the topic. Newton and Leibniz would now do just that; they added to and elaborated upon the subject of Calculus during the last third of the 17th century [5, p. 608]. These two men essentially "invented the general concepts of derivative ('fluxion', 'differential') and integral ….recognized differentiation and integration as inverse operations…developed algorithms to make calculus the powerful computational instrument it is," and most importantly, "While in the past the techniques of calculus were applied mainly to polynomials, often only of low degree, they were now applicable to 'all' functions, algebraic and transcendental" [9, p. 142].

Isaac Newton was born prematurely on Christmas Day of 1642. After surviving his initially dangerous situation, the death of his father, and being abandoned by his mother [2, p.160], Newton grew up to become one of the greatest mathematicians of the seventeenth century and perhaps of all time. In the words of his contemporary, Leibniz, "Taking mathematics from the beginning of the world to the time of Newton, what he has done is much the better half" [1, p. 391]. In fact, Newton was so well-respected and admired that Isaac Barrow, his only predecessor as the esteemed Lucasian chair at Cambridge, actually resigned the position in 1669 in his favor [4, p. 301]. Among Newton's accomplishments were the generalized binomial theorem, discovered early in 1665, a theory of colors, and of course, his development of Calculus in 1666 [2, p. 164].

Newton's development of Calculus was based on the fluent, or any quantity that constantly changed, which could be either geometric or physical [1, p. 395]. His theory was centered around the fluxion—a rate of change, principal fluxion—the constant rate of increase of any given fluent, and the moment of a fluent—the infinitely small amount that the fluent of x changes in a small time o, denoted $\dot{x}o$ [4, pp. 305-306]. He described a limit as an "ultimate ratio," which he viewed as the amount of a vanishing number, o, just before it ceased to exist [5, p. 612]. As his theories progressed, Newton was able to make a more clear definition, which if translated into algebraic terms, would have been very close to the modern definition of a limit [8, p. 520]. The definition, in Newton's own words, can be stated as follows: "Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given distance, become ultimately equal" [1, p. 398]. Here, the statement "nearer to each other than any given distance" is very close to the modern day $\varepsilon - \partial$ definition of a limit, as Newton was positing that a quantity must come within ε of its limit, no matter what ε was, without ever algebraically making this connection.

Newton wrote three accounts dealing with his development of Calculus, none of which was published until several years after its creation. These separate issues were written perhaps as a means of constantly improving upon the subject, or Newton may have thought of each edition as written for different reasons—"to derive results effectively, to supply useful algorithms, or to give convincing proofs" [9, p. 143]. The first account, *De analysi per aequationes numero terminorum infinitas (On Analysis by Equations with Infinitely Many Terms)*, was written in 1669 but not published until 1711. In this early work, Newton was already capable of determining the area under the curve $y = ax^{m/n}$ for any m and n to be $\frac{ax^{(m/n)+1}}{(m/n)+1}$ by using the results of his binomial theorem [1, pp. 395-396].

In Newton's second text, *De Methodis Serierum et Fluxionum (A Treatise on the Methods of Series and Fluxions)* (1671), he was able to determine that a maximum or minimum for a curve existed only when its fluxion, or derivative, was equal to zero. As quoted by Edwards [3, p. 209], Newton had the following to say on the subject:

> 'When a quantity is greatest or least, at that moment its flow neither increases nor decreases: for if it increases, that proves that it was less and will at once be greater than it now is, and conversely so if it decreases. Therefore seek its fluxion [by previously described methods] and set it equal to nothing.'

Newton was logically describing the first derivative test as it became known in modern times: A function reaches its maximum and/or minimum value whenever its derivative is equal to zero.

In Newton's third account of Calculus, *De quadratura curvarum (On the Quadrature of Curves)*, he attempted to avoid the infinite by using prime numbers and prime ratios [1, p. 397]. This text was either written in 1676 [1, p. 397] or between 1691 and 1693 and was much more technical than either of his two previous attempts at creating Calculus [3, p.

226]. Newton stated the following formula for the area under the curve $y = x^{\theta}(e + fx^{\eta})^{\lambda}$ [3, p. 227]:

$$\text{Area under the curve} = Q\left[\frac{x^{\pi}}{s} - \frac{r-1}{s-1}\frac{eA}{fx^{\eta}} + \frac{r-2}{s-2}\frac{eB}{fx^{\eta}} - \frac{r-3}{s-3}\frac{eC}{fx^{\eta}} + \ldots\right],$$

where $Q = \frac{(e + fx^{\eta})^{\lambda+1}}{\eta f}$, $r = \frac{\theta+1}{\eta}$, $s = \lambda + r$, $\pi = \eta(r-1)$

and A, B, and C denoted their immediate predecessors. That is,

$A = \frac{x^{\pi}}{s}$, $B = -\frac{r-1}{s-1}\frac{eA}{fx^{\eta}}$, $C = \frac{r-2}{s-2}\frac{eB}{fx^{\eta}}$

It is important to note that this particular formula is capable of computing something as simple as $\int x^n dx$ or as complicated as $\int \frac{x}{1-2x^2+x^4}$ dx.

Newton was able to discover the fundamental theorem of Calculus using his methods of fluxions and inverse fluxions in 1666. His computation of area through anti-differentiation was the first appearance of the fundamental theorem of Calculus in its precise form [3, p. 196]. Newton had often thought of curves as generated by the motion of the variables x and y. Therefore, he found it to be completely obvious that $\dot{A} = y\dot{x}$, or $\frac{\dot{A}}{\dot{x}} = y$, where A is the area under the curve generated by x and y [8, p. 514].

These were not the only accomplishments made by Newton in his invention of Calculus, nor were they the only attempts to derive Calculus concepts during Newton's lifetime. Gottfried Wilhelm Leibniz was born in 1646 at Leipzig, and by the time he was fifteen, he was ready to enter the University of Leipzig. Leibniz studied a number of subjects, including theology, law, and mathematics. By the time he was twenty, he was ready for his doctorate in law. Sadly, Leibniz was refused this honor due to his young age, but he would continue to excel throughout his academic career and hold many prestigious positions in both the government and in the academic world [1, p. 400], [2, pp. 184-185].

By the end of 1676, Leibniz had discovered the solutions to many of the problems that Newton worked on. That is, Leibniz was able to put his genius to work to create his own development of Calculus [2, p. 187]. Unlike Newton, Leibniz did not define Calculus concepts in terms of fluents and fluxions. Instead, his main focus was the differential. Leibniz thought of a curve as being a polygon with an infinite number of sides of infinitely small magnitude, rather than as determined by motion the way that Newton had [9, p. 146]. Leibniz realized in 1673 that the tangent to any given curve was dependent on the difference between y-coordinates in a ratio with the difference between x-coordinates as such differences became infinitely small [1, p. 402]. Leibniz called these respective differences the differential of x, denoted by dx, and the differential of y, denoted by dy. That is dx $= x_i - x_{i-1}$, dy $= y_i - y_{i-1}$, and their ratio, $\frac{dy}{dx} = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}$, which Leibniz called the differential quotient, was the necessary quantity needed to find the tangent line [9, p. 146].

In the 1684 text, *Nova methodus pro maximus et minimus, itemque tangetibus, qua nec irrationals quantitates moratur (A New Method for Maxima and Minima, and also for Tangents, which is not Obstructed by Irrational Quantities)*, Leibniz was able to easily derive the product, power, and quotient rules for derivatives, as they are known today [1, p. 404]. The rule for finding the n^{th} derivative of a product is, in fact, still known as Leibniz's rule to this day [4, p. 310]. As an example of Leibniz's proofs of various differentiation rules, consider the quotient rule, namely $d\frac{y}{x} = \frac{xdy - ydx}{x^2}$. Leibniz wrote the following as his proof of the result [3, p. 256]:

$$d\frac{y}{x} = \frac{y+dy}{x+dx} - \frac{y}{x}$$

$$= \frac{x(y+dy) - y(x+dx)}{x(x+dx)}$$

$$= \frac{xy + xdy - xy - ydx}{x^2 + xdx}$$

$$= \frac{xdy - ydx}{x^2 + xdx}$$

Since xdx is assumed to be very small, the quantity is equal to

$$\frac{xdy - ydx}{x^2}.$$

Note that Leibniz's formulation was completely independent of the nature of y and x, whereas much of Newton's information was given in terms of examples that could be easily generalized. Furthermore, Leibniz's notation, which was a major focus for him, was often that which led his theories to be so easily proven. This was radically different from Newton's approach, as Newton was not as interested in notation as was Leibniz [3, p. 265-266].

For Leibniz, the integral was "an infinite sum of infinitesimal rectangles with base dx and height y," and the leftover triangular area was actually his link to differentiation, as it involved both a base of dx and a height of dy [9, p. 147]. Much like his use of the modern dx and dy with differentiation, Leibniz also introduced the modern-day symbol for integration, the elongated S, $\int$, to denote his summation of rectangles for integrals [1, p. 403]. This symbol was first printed in Leibniz's *Acta Eruditorum* of 1686 [3, p. 260].

Using the product rule for derivatives, Leibniz was able to arrive at the formula for integration by parts using only differentials, much in the way that modern Calculus students learn the formula. The formulation is as follows [9. p. 148]:

Given the product rule, d(xy) = xdy + ydx, it follows that

$$\int d(xy) = \int xdy + ydx = \int xdy + \int ydx$$

Then, given the inverse relationship between integrals and derivatives,
$$xy = \int xdy + \int ydx \Rightarrow \int xdy = xy - \int ydx$$

This formulation relied on knowledge that the derivative and the integral were inverses, which Leibniz had come to accept, as he had noticed that integrals depended on sums, while derivatives depended on differences. Leibniz deduced that there must be the same inverse relationship between derivatives and integrals as there was between sums and differences [1, p. 402].

The fundamental theorem of Calculus first appeared in Leibniz's *Acta Eruditorum* of 1686 in the following form [3, p. 257-258]:

> Given a curve with ordinate z, whose area is sought, suppose it is possible to find a curve with ordinate y such that
> $\frac{dy}{dx} = \frac{z}{c}$, where a is a constant
> Then, $zdx = cdy$, so the area is given by
> $\int zdx = \int cdy = c\int dy = cy$ (assuming the curves pass through the origin)
> Let c=1. Then, subtract the area over [0, a] to that over [0, b] to obtain
> $$\int_a^b zdx = y(b) - y(a)$$

When Leibniz published his results, Newton's supporters cried "foul!" and accused him of plagiarism. Leibniz had seen some of Newton's documents in a 1673 visit to London, and there had also been several letters sent between Newton and Leibniz on the subject. Newton's work was published so late that, while he had actually begun to develop his method of fluxions long before Leibniz began working with differentials, many of Leibniz's supporters actually accused Newton of plagiarism, as well [2, pp. 187-188]. At the time that Leibniz may have seen Newton's early work on *De analysi*, however, it is unlikely that Leibniz would have been able to understand the document very well. At that point in his mathematical career, Leibniz was not very well-trained in Newton's area of analysis [1, p. 400].

In the early 1670's, Newton had sent a letter to Leibniz discussing integration using the binomial theorem. However, only a few months after the first letter was sent, when Leibniz was intrigued and wanted to learn more, Newton sent a completely encrypted letter back to his rival that was supposed to have been about fluxions. This poses the question as to why Newton would have offered to share the information in the first place if he was merely going to hide his results [5, p. 512]. When the first edition of Newton's 1687 *Principia* was published, Newton admitted to collaboration, stating that Leibniz had come to similar results as himself. By the time the third edition was printed in 1726, the reference was completely removed [1, p. 399].

In 1712, the Royal Society of London, of which Newton was president at the time, found Leibniz to be guilty of plagiarism [3, p. 267]. The main evidence used against Leibniz was that it was possible for him to have seen Newton's papers on his short visits to England during 1673 and 1676, which as stated, probably would not have been understandable at the time anyway [5, p. 515]. Leibniz was given virtually no chance to plead his case or to submit his work to show the glaring differences between his formulation of Calculus and Newton's own. As a result of the controversy and bitterness on both sides, Newton's followers were virtually shut off from the next century of mathematical progress [3, p. 267]. Furthermore, by the time Leibniz died in 1716, his status was almost completely diminished. While Newton's death had been given a great reception in England, accounts have dictated that Leibniz's burial was so ignored that only one faithful servant was in attendance [2, p. 190].

After both men were affected so harshly, and the history of English mathematics was perhaps significantly altered, most modern historians would argue that Calculus was developed simultaneously but separately by both Newton and Leibniz [2, p. 188]. Any comparison of their writings shows that the two men found their motivations in completely different areas. Their methods and notation were also radically different. Furthermore, Newton's original admission in the *Principia* that Leibniz had come to similar conclusions implies that there were no original evil intentions on either part. Nevertheless, by the end of the 1670's, a strong foundation for Calculus had been built.

Once Calculus was "invented," it was now possible for mathematicians to spread and expand upon the ideas that Newton and Leibniz had presented. Johann Bernoulli, one of Leibniz's most famous supporters, aided the French Marquis de L'Hospital in his goal to learn the new and exciting Calculus [2, p. 191]. By signing a pact in 1692, Bernoulli had agreed to send any pertinent discoveries in the subject to L'Hospital, as well as keep the information secret from everyone else. Bernoulli was offered "a large monthly" salary for his contributions. In 1696, L'Hospital published a collection of these works in the first Calculus textbook, *Analyse des infiniment petits pour l'intelligence des lignes courbes (Anlaysis of infinitely small quantities for the understanding of curves)* [8, p. 532]. One of the most well-known results included in the book, known as L'Hospital's rule, was actually Bernoulli's [1, p. 420]. The rule states: $f(a) = g(a) = 0$ and both f and g are differentiable $\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$. The rule can also be used for the indeterminate form of $\frac{\infty}{\infty}$. As an example of how the rule works, consider the function

$$f(x) = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}.$$

Observe: $f(a)=\dfrac{\sqrt{2a^3(a)-a^4}-a\sqrt[3]{a^2(a)}}{a-\sqrt[4]{a(a^3)}}=\dfrac{\sqrt{a^4}-a\sqrt[3]{a^3}}{a-\sqrt[4]{a^4}}=\dfrac{a^2-a(a)}{a-a}=\dfrac{a^2-a^2}{a-a}=\dfrac{0}{0}$

Then, applying "L'Hospital's Rule,"

$$\lim_{x\to a} f(x)=\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}$$

$$=\lim_{x\to a}\frac{\frac{1}{2}\left(2ax^3-x^4\right)^{-1/2}\left(2a^3-4x^3\right)-a\left(\frac{1}{3}\right)\left(a^2x\right)^{-2/3}a^2}{\frac{1}{4}\left(ax^3\right)^{-3/4}\left(3ax^2\right)}$$

$$=\frac{\frac{1}{2}\left(\frac{1}{\sqrt{2a^4-a^4}}\right)\left(2a^3-4a^3\right)-\frac{1}{3}a^3\frac{1}{\left(\sqrt[3]{a^3}\right)^2}}{\frac{-3}{4}a^3\frac{1}{\left(\sqrt[4]{a^4}\right)^3}}$$

$$=\frac{\frac{1}{2\sqrt{a^4}}\left(-2a^3\right)-\frac{1}{3}a^3\frac{1}{a^2}}{\frac{-3}{4}a^3\frac{1}{a^3}}=\frac{\frac{1}{2a^2}(-2a^3)-\frac{1}{3}a}{\frac{-3}{4}}=\frac{-a-\frac{a}{3}}{\frac{-3}{4}}=\frac{\frac{-4a}{3}}{\frac{-3}{4}}=\frac{16a}{9}$$

Although Newton and Leibniz had developed a strong enough topic to include advances like the rule that L'Hospital was credited with, many mathematicians were uneasy with the fluxions and differentials due to the bizarre concept of infinitesimals that were zero but not zero at the same time. In particular, Joseph Louis Lagrange suggested in 1784 that the Berlin Academy offer a prize to anyone who could successfully address the question of the infinite. At this point, Lagrange had realized that "..the foundations of the calculus were unsatisfactory.." as a result of a lack of clear understanding of the infinitesimals used in both Newton's method of fluxions and Leibniz's work with differentials. When the competition yielded no satisfactory results, Lagrange took it upon himself to attempt to find a more rigorous foundation for Calculus [6, pp. 40-42].

In his 1797 *Thèorie des functions analytiques*, Lagrange attempted to remove the concepts of limits and infinitesimals from Calculus by defining the derivative in terms of "derived functions." Lagrange's main goal in making this attempt was to make Calculus more logical [1, p. 489]. His derived functions came from the following process [3, pp. 296-297]:

> Begin with the power series expansion of f(x). Then, "by the theory of series,"
> $f(x+i)=f(x)+pi+qi^2+ri^3+\ldots$
> Lagrange claimed that $p(x)=f'(x)$

To find the coefficients, write $i = i + o$, and observe

$$f(x+i+o) = f(x) + p \cdot (i+o) + q \cdot (i+o)^2 + r \cdot (i+o)^3 + \ldots$$

$$= f(x) + pi + po + q(i^2 + 2io + o^2) + ri^3 + 3ri^2o + 3rio^2 + ro^3 + \ldots$$

$$= f(x) + pi + qi^2 + ri^3 + si^4 + \ldots + po + 2qio + 3ri^2o + 4si^3o + \ldots$$

Next, replace x by x + o to obtain the following

$$f(x+i+o) = f(x+o) + p(x+o)i + q(x+o)i^2 + r(x+o)i^3 + \ldots$$

$$= [f(x) + f'(x)o + \ldots] + [p(x) + p'(x)o + \ldots]i + [q(x) + q'(x)o + \ldots]i^2 + \ldots$$

$$= f(x) + pi + qi^2 + ri^3 + \ldots + p'io + q'i^2o + r'i^3o + \ldots$$

Comparing coefficients in both representations yields

$$q(x) = \frac{1}{2}p'(x) = \frac{1}{2}f''(x)$$

$$r(x) = \frac{1}{3}q'(x) = \frac{1}{3!}f'''(x)$$

$$s(x) = \frac{1}{4}r'(x) = \frac{1}{4!}f''''(x)$$

and so on.

Therefore, $f(x+i) = f(x) + f'(x)i + \frac{f''(x)}{2!}i^2 + \frac{f'''(x)}{3!}i^3 + \ldots$, that is, Taylor's series.

Then, Lagrange posited that the derived functions f', f'' and so on coincided with the function's derivatives, and very little knowledge of Calculus was needed to come to this result. Hence, Lagrange had defined the derivative of a function in terms of the coefficients in the power-series of f, translated appropriately [9, p. 158].

Although Lagrange had made a successful attempt at formulating the derivative for a special type of functions, his proof contained errors that prevented him from completely removing infinitesimals from Calculus. A series expansion was always possible, and his work begged the question of whether the series did, in fact, converge [1, p. 489]. Despite Lagrange's blunder in this particular endeavor, he was successful with many other contributions to the theory of Calculus, including to the field of differential equations [4, p. 335]. Among these accomplishments was the "Lagrange property of the derivative," which can be stated as follows:

$f(x+i) = f(x) + if'(x) + iV$, where V goes to zero with I, which means given any D, *i* can be chosen sufficiently small so that V is between –D and D, or

$$i[f'(x) - D] < f(x+i) - f(x) < i(f'(x) + D)$$

This property was used “to derive many of the known results about functions and their derivatives, including the properties of maxima and minima, tangents, areas, arc lengths, and orders of contact between curves” and relied heavily on Lagrange’s Taylor series expansion of the function f(x +i) [6, p. 116]. Also, he can be credited with realizing, perhaps for the first time, that the derivative of a function was also a function [9, p. 159].

One of the next important breakthroughs to come in the development of Calculus concepts came from the secluded Bohemian priest, Bernard Bolzano. In his pamphlet, *Purely analytical proof of the theorem, that between each two roots which guanrantee an opposing result, at least one real root of the equation lies,* Bolzano gave one of the first good definitions of continuity. Quoted by Edwards [3, p. 308], Bolzano stated that continuity could not be explained without understanding the following:

> ‘A function f(x) varies according to the law of continuity for all values of x which lie inside or outside of certain limits, is nothing other than this: If x is any such value, the difference $f(x+\omega)-f(x)$ can be made smaller than any given quantity, if one makes ω as small as one wishes.’

Hence, for f to be continuous on an interval, I, $\lim_{\omega \to 0} f(x+\omega)-f(x)=f(x)$, $\forall x \in I$.

While Bolzano’s results were not very widely circulated, Augustin-Louis Cauchy stated a very similar result in his 1821 *Cours d’analyse*. His definition of continuity reads as follows:

> ‘The function f(x) will be a continuous function of the variable x between two assigned limits [“limit here means “bound”] if, for each value of x between those limits, the numerical [absolute] value of the difference $f(x+a)-f(x)$ decreases indefinitely with a. In other words, the function f(x) is continuous with respect to x between the given limits if, between these limits, an infinitely small increment in the variable always produces an infinitely small increment in the function itself’ [6, p. 87].

After clearly defining what it meant for a function to be continuous, the next obvious question to ask was what conditions guaranteed a discontinuity at a point. Clearly, contradicting the above definition was means enough, but Cauchy was able to clearly define the idea in functional notation as early as 1814 in the following way:

> Let $\Phi(z)$ be a given function, Z denote the point in question, and let ξ be some tiny amount. Then, there is a jump discontinuity at Z if $\lim_{\xi \to 0} f(Z+\xi)-f(Z-\xi) \neq 0$ [6, p. 94]

While he did not explicitly state this result in limit notation, it was obvious that Cauchy had the idea in mind when he described jumps in terms of the functional values unexpectedly going from one point to another.

One of Cauchy's most notable contributions was the clarification of concept of limit and its establishment as the basis for the other main ideas of Calculus [9, p. 160]. In addition to *Cours d'analyse*, Cauchy wrote two more major texts. These were his *Resume des leçons sur le calcului infinitesimal* of 1822 and *Leçons sur le calcul differential* of 1829. All three of Cauchy's texts were some of the first to set rigor as a goal [3, p. 304]. He is believed to have been the first rigorous definitions of Calculus terms using the modern epsilon-delta notation, but his definitions actually sound more intuitive than algebraic [6, pp. 6-7]. Despite this seeming discrepancy, Cauchy did indeed use epsilons and deltas in many of his proofs [9, p. 162].

Another famous contribution was the 1821 idea of showing that the terms of a sequence get closer to one another was enough to show convergence. This result, now known as the "Cauchy criterion," states that, "a sequence $\{s_n\}$ of real numbers converges if and only if it is a Cauchy sequence." Cauchy himself was unable to prove a one-sided implication, that Cauchy sequences must be convergent, because he did not have enough knowledge about real and irrational numbers [7, pp. 175-76]. In modern day notation,

> A sequence $\{x_n\}$ is called a *Cauchy sequence* whenever $\forall \varepsilon > 0, \exists N \in \mathrm{N}$ (depending, in general, upon ε such that for each $m \geq N$ and $n \geq N, |x_n - x_m| < \varepsilon$ [10, p. 67].

In addition to Cauchy's advances with limits and continuity, he was also arguably the first person to see the integral as a limit of sums, rather than simply as an inverse of the derivative [4, p. 364]. He wrote that $\int_a^b f(x)dx = \lim_{\partial \to 0} \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$. As a result of this improved definition, the fundamental theorem of Calculus could now be proven without forcing the mathematician to view the integral as an area [9, p. 163].

Bernhard Riemann, whose name is attached to the modern-day integral, drew off of Cauchy's work when creating his definitions. There are two glaring differences between the conceptions of these two men when considering the subject of integration: 1) When Cauchy defined his integral in terms of sums, he used only the endpoints of intervals, while Riemann showed it was possible to use any arbitrary point, and 2) Cauchy assumed that functions must be continuous in order to be integrable, whereas Riemann was able to prove otherwise [6, p. 162].

In Riemann's 1854 paper, "Habilitationschrift," devoted to representing functions in terms of trigonometric series, section 4 began with the question of what it meant for a function to be integrable. Riemann then gave the following immediate answer [3, pp. 323-4]:

> 'In order to establish this, we take of sequence of values $x_1, x_2, ..., x_{n-1}$ lying between a and b and ordered by size, and for brevity, denote

$x_1 - a$ by ∂_1, $x_2 - x_1$ by ∂_2, …, $b - x_{n-1}$ by ∂_n, and proper positive fractions by ε_i. Then the value of the sum

$$S = \partial_1 f(a + \varepsilon_1\partial_1) + \partial_2 f(x_1 + \varepsilon_2\partial_2) + \partial_3 f(x_2 + \varepsilon_3\partial_3) + ... + \partial_n f(x_{n-1} + \varepsilon_n\partial_n)$$

will depend on the choice of the intervals ∂_i and the quantities ε_i. If it has the property that, however the ∂_i and the ε_i may be chosen, it tends to a fixed limit A as soon as all the ∂_i become infinitely small, then this value is called $\int_a^b f(x)dx$. If it does not have this property, then $\int_a^b f(x)dx$ is meaningless.'

Thus, Riemann chooses an *arbitrary* point $\overline{x_i} = x_{i-1} + \varepsilon_i\partial_i$ in the i^th^ subinterval $[x_{i-1} - x_i]$ of his partition i =1, …, n, and defines the integral by $\int_a^b f(x)dx = \lim_{\partial \to 0} \sum_{i=1}^{n} f(\overline{x_i})(x_i - x_{i-1})$, where δ denotes the maximum of the lengths ∂_i of the subintervals of the partition of [a,b] [3, p. 323].

After providing a clear concept of what an integral was, Riemann decided to investigate its strength. While mathematicians had assumed that the integral was only defined for a continuous function, Riemann was able to use the following counterexample to show that the assumption was actually false in 1854:

Let $f(x) = \sum_{n=1}^{\infty} \frac{\mathbf{B}(nx)}{n^2}$, where $B(x) = \begin{Bmatrix} x - \langle x \rangle, x \neq k/2 \\ 0, x = k/2 \end{Bmatrix}$. The function is discontinuous at $x = \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}, ...$ nevertheless, the series converges uniformly and the functions $f_n(x)$ are integrable so that f(x) is integrable, as well [7, p. 230].

Furthermore, Riemann was able to use his new theory on integration in order to prove various results about Fourier series, which represents functions in terms of trigonometric series, which was Riemann's original goal [8, p. 726].

Clearly, Riemann's development of the integral and his results on integrability differ in extreme measures from Newton's inverse fluxions, as well as from Leibniz's early integration results. However, without their initial developments, works like Riemann's and Cauchy's would have to have been derived from absolutely nothing and may never have been produced. It is in this way that the development of Calculus can be viewed as the collaborative efforts of mathematicians over the span of hundreds of years—

thousands of years if one wishes to include the works of Eudoxus and Archimedes within the scope of Calculus.

In conclusion, while Sir Isaac Newton and Gottfried Wilhelm Leibniz are credited as being the co-inventors of Calculus, it is necessary to view the development of Calculus as the work of many mathematicians. Newton and Leibniz were the first to axiomatically define the subject; others would systematically improve upon their works in order to make the concepts of Calculus more clearly defined. After centuries of writing and re-writing the definitions of limit, continuity, derivative, and integral, Calculus became a rich, rigorous subject of high importance in mathematics. Without Newton, Leibniz, Lagrange, Cauchy, Riemann, and many, many other mathematicians many useful problems would still be solvable through only case-by-case examples, like those of Sluse and Fermat. Perhaps, some would even be completely unsolvable.

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