

Curvature of Surfaces in 3-Space

History of the Study of Curvature

Curvature has ultimately had a single role throughout the history of mathematics: to illustrate the natural beauty of mathematics and to describe, in the best way, the mathematical aspects of nature. The notion of curvature first began with the discovery and refinement of the principles of geometry by the ancient Greeks circa 800-600 BCE. Curvature was originally defined as a property of the two classical Greek curves, the line and the circle. It was noted that lines do not curve, and that every point on a circle curves the same amount. The actual study of curvature began when Aristotle expanded upon these two points and declared that there are three kinds of loci: straight, circular, and mixed. It was from this premise that the true study of curvature began.

Apollonius of Perga devised methods for calculating the radius of curvature in the 3rd century BCE. These methods were similar to those of Huygens and Newton (discovered some 2000 years later), but neither Apollonius nor his contemporaries were able to expand on them since their methods of exhaustion proved to be too rigorous. This helped to push the study of curvature further along as there was more research to be done. (Margalit, *The History of Curvature*, 2005)

The next momentous advancement in the study of curvature came from Nicole Oresme in the fourteenth century CE. Oresme was the first person to hint at an actual definition of curvature. He also assumed that there was a specific measure of twist which he called "curvitas." By observing multiple curves at once, Oresme eventually proposed that the curvature of a circle proportional to the multiplicative inverse of its radius. This would eventually provide the driving force behind the quest of finding the curvature of a general curve, a measurement that could be applied to any curve. (Margalit, *The History of Curvature*, 2005)

Johannes Kepler (1571-1630) made the next contribution to the notion of curvature. While working on the problem of Al Hazin, (finding the image of a brilliant point when reflected off of a circle), Kepler arrived at the notion of using a circle to measure the general curvature of the curve at the point of reflection. This approximating circle would come to be known as a curve's "circle of curvature" at a point. The radius of the circle is inversely proportional to the extent to which the curve bends at that point. The circle of curvature was crucial to the development of curvature because it marked the first attempt to truly measure the degree of curvature, the measure of how much a curve twists. (Margalit, *The History of Curvature*, 2005)

Rene Descartes and Pierre de Fermat were the first to express general curves in geometry as equations. This was a step towards the role curves would play in Calculus, but Descartes' and Fermat's work on the subject was incomplete because the analyses lacked any mention of pi. As a result, the development of analytic geometry was stunted for the start of the seventeenth century. However, change came in 1673 when a mathematician named Christiaan Huygens published the influential book *Horologium*

oscillatorium sive de motu pendulorum ad horologia aptato demonstrationes geometrica. (Margalit, The History of Curvature, 2005)

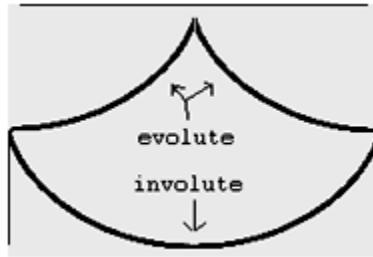


Figure 1

In his text Huygens described two more features of a curve, its evolute and involute which are illustrated in Figure 1 (Margalit, cyccyc.gif). The involute is created in a series of movements, and it is important to notice that at each step, the string is tangent to the evolute and perpendicular to the involute. By this, Huygens would eventually define the radius of curvature of the involute as the distance between the points of contact between the involute and evolute with the string.

This bears significance because of its relevance to what the working definition of curvature eventually became. However, Huygens' method was flawed: in order to find the radius of curvature, the evolute had to be provided, and as a result, the theory was useless for measuring arbitrary curves. (Margalit, The History of Curvature, 2005)

Calculus was finally invented in the late 17th century. Calculus's ability to deal with limits and infinitesimal amounts helped the study of curvature. A curve can have a different curvature at every point, so mathematicians needed a way to view an infinitely small section of a curve in order to measure its curvature at that point. The modern method of measuring curvature is accredited to one of the co-founders of Calculus, Sir Isaac Newton.

To Sir Isaac Newton, a curve was an object of beauty. Newton viewed curvature as its own classification of science, and therefore scrutinized its every aspect. In his work, *Methods of Series and Fluxions*, Newton proposed to measure the curvature of any curve at a given point. He noted that this process required a certain...elegance. Newton began his process by first observing the three basic properties of curves: A circle has a constant curvature which is inversely proportional to its radius; the largest circle that is tangent to a curve (on its concave side) at a point has the same curvature as the curve at that point; and the center of this circle is the "centre of curvature" of the curve at that point. (Margalit, The History of Curvature, 2005)

Newton's definition of the center of curvature was momentous because it was in his work on this subject that he first introduced the concept of infinitesimals, actually implementing calculus. He stated that the center of curvature "is the meet of normals at indefinitely small distances from [the point in question] on its either side." It was from this that Newton would formulate his equation for the radius of curvature, and eventually modify that equation to be used in polar coordinates as well. There was however a flaw in Newton's equations - they yielded "undefined" solutions at points of inflection. (Margalit, The History of Curvature, 2005)

From Newton's observations and from the properties of calculus, it is known that curves behave like straight lines near a point of inflection. From this, Newton theorized that since the radius of curvature of a straight line is infinite, the radius of curvature at points of inflection is also infinite. From this Newton calculated the formulae for the radii of curvature of several curves, including the cycloid and the Archimedean spiral. These calculations were notable in that they were performed analytically through

the use of calculus. Until Newton co-invented calculus, the radius of curvature, and curvature itself was calculated by extraneous geometrical methods. It was Newton who first calculated a value for curvature without using geometry.

The next mathematician to have historic effect on curvature was Leonhard Euler, who made revolutionary statements about curvature in 1774. Euler devised a new way of defining curvature. He defined curvature as $\frac{d\theta}{ds}$, which is the change in angle of the tangent divided by the change in arc length. This only applied to an infinitely small location on the curve. Euler was the mathematician responsible for the important theorem that the magnitude of curvature equals the magnitude of the second derivative of a parameterization of the curve at a specific point. (Margalit, *The History of Curvature*, 2005)

Generally speaking, there are two important types of curvature: extrinsic curvature and intrinsic curvature. The definition of curvature has been modified throughout history and it changes minutely depending upon how many dimensions are being observed as well as on what specific curve is involved. Curvature, defined in 3-space, is the measure of how much the curve “bends” at a single point. This can be thought of as the rate of change of the angle formed between the tangent and the curve as the tangent is drawn along the curve. The discussion thus far has concerned extrinsic curvature in two- and three-space throughout history. This curvature describes a space curve (defined as a curve which may pass through any region of three-dimensional space) entirely in terms of its torsion (the rate of change of the osculating plane) and the initial starting point and direction. (Weisstein, *Curvature*)

Exploration of intrinsic curvature developed after the study of the extrinsic. The main types of curvature that emerged from this were mean curvature and Gaussian curvature. Mean curvature was the relevant to applications of the time and was, as a result, the most studied. Gauss was the first to recognize the importance of the Gaussian curvature. Gauss said that because Gaussian curvature is “intrinsic,” it is detectable to hypothetical two-dimensional “inhabitants” of the surface. The importance of Gaussian curvature derives from an inhabitant’s control over the surface area of spheres around himself. (Weisstein, *Curvature*)

Gaussian curvature is regarded as an intrinsic property of space that is independent of the coordinate system that is used to describe that space. If there exists a surface in three-space, at a specific point, there is a plane tangent to that surface. A generalization of curvature known as normal section curvature can be computed for all directions of that tangent plane. From calculating all the directions, a maximum and a minimum value are obtained. The Gaussian curvature is the product of those values. The Gaussian curvature signifies a peak, a valley, or a saddle point, depending on the sign. If positive, a valley or peak, if negative, a saddle point, and if the Gaussian curvature is zero, then the surface is flat in at least one direction. (Weisstein, *Curvature*)

A modern day application of curvature can be found in the study of modern physics. In relativity, one concept of discussion concerns how different elements of the universe affect light. The application of curvature is best described by John Wheeler, an American theoretical physicist: “Space tells matter how to move and matter tells space how to curve.” For this application, it is best to visualize part of the

universe as an infinitesimally thin bed sheet. In the study of the quantized theory of light, it is learned that mass creates a localized distortion in the space time continuum. To expand this to the bed-sheet metaphor, consider stretching the sheet (representing space-time) in every direction and then placing a bowling ball on top of it. The sheet sags down at the point where the bowling ball rests. This is how mass distorts space-time.

When mass distorts space time, light traveling in that vicinity is bent, meaning the path of the light is changed. It is important to measure how much the light is bending, and this is where curvature comes into play. The light bends because the mass creates a gravitational force. When light is affected this way by gravity its frequency shifts towards the red end of the spectrum, a phenomenon called gravitational red-shift. The size of the mass determines the extent to which space-time is distorted; the larger the mass, the more space time is curved, as illustrated in Figure II (Carroll & Ostlie, 1996). Thus, the path of the light depends upon the amount of curvature on the path. Therefore the curvature affects the extent to which the light shifts.

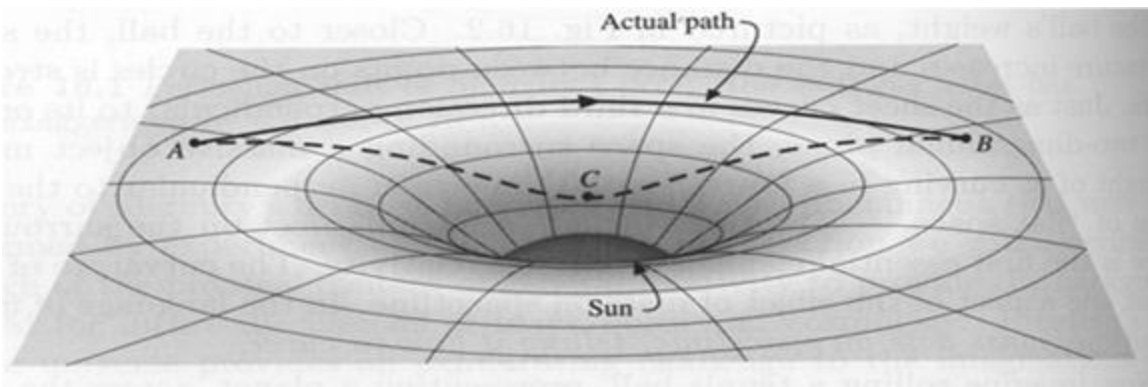


Figure II

Properties of Space Curves

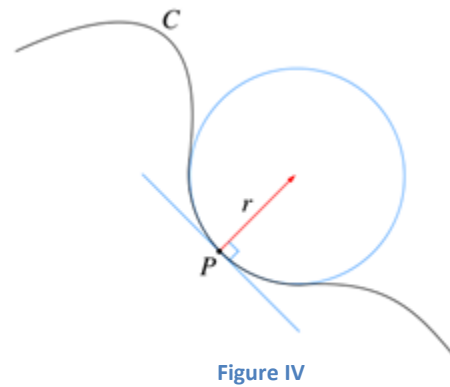
Part one of this project explored space curves: their vector equations of points, normals, tangents, and binormals, and their qualities of torsion (τ) and curvature (κ). A summary of these follows.

Where $\vec{r}(s)$ describes the position of a point on a curve parameterized in terms of arc length s and $\vec{r}' = \frac{d\vec{r}}{ds}$, $\vec{r}'' = \frac{d^2\vec{r}}{ds^2}$, $\vec{v} = \frac{d\vec{r}}{dt}$, and \hat{T} , \hat{N} , and \hat{B} are the unit tangent vector, the principle normal vector, and the binormal vector respectively with: $\hat{T} = \frac{\vec{v}}{\|\vec{v}\|}$, $\hat{N} = \frac{\vec{T}'}{\|\vec{T}'\|}$, and $\hat{B} = \hat{T} \times \hat{N}$, the following equations result (Weisstein, Curvature):

$$\tau = -\hat{N} \cdot \hat{B}'$$

$$\kappa = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$$

These mathematical definitions will serve as tools in building a further understanding of curvature of surfaces in 3-space, but, first, the concepts they describe must be further explored. Curvature can be most easily visualized as being related to the radius of the osculating circle that most closely fits the curve at that point, as can be seen in Figure IV (User:Cepheus, 2006). Torsion, on the other hand, can be imagined as the rate of rotation of the osculating plane described by the mutually orthogonal tangents, normals and binormals of the curve at each point. This concept is more difficult to visualize with a static image, but it is illustrated in Figure IV (Schmies, 2007).



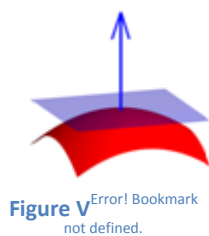
Mathematically, the radius of curvature ρ can be given by (Seggern, 1993):

$$\rho = \left| \frac{ds}{d\theta} \right|$$

Where ds is the differential of the arclength along the curve path and θ refers to the angle of the tangent with the x-axis which changes its direction over ds by an angle of $d\theta$. Radius of curvature can also be expressed in terms of the derivatives of the curve. For example, for the implicitly defined curve $f(x, y) = 0$ (Seggern, 1993):

$$\rho = \frac{(f_x^2 + f_y^2)^{3/2}}{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}$$

Principal Curvature



The tools developed to explore space curves evolve naturally into tools that can be used to explore curved surfaces. The behaviors of the tangent lines from point to point on the curve that was so useful in describing torsion and curvature are comparable to the tangent planes that are so integral to exploring surfaces. All tangent lines to a point on a surface will fall in the same plane: the tangent plane to

the surface at that point. Any curve embedded in the surface that passes through that point will have a tangent at that point which falls in this tangent plane. So, if we take a normal vector perpendicular to the tangent plane at this point (Figure V (Alexandrov)), this normal vector at point (x_0, y_0) on surface

$$z = f(x, y) \text{ would be given by } \hat{N} = \begin{pmatrix} f_x(x_0, y_0) \\ f_y(x_0, y_0) \\ -1 \end{pmatrix}.$$

A plane containing the unit normal, \bar{u} , and the tangent vector, \hat{T} , for a point on the surface intersects with that surface along a curve with a normal curvature, k_n , shared by all curves with the same tangent vector at that point. The maximum and minimum of these normal curvatures at a point are called the principal curvatures, k_1 and k_2 , and they measure the maximum and minimum “bending” of the surface at that point. Figure VI (Gaba, 2006) illustrates the principal curvatures of a saddle surface.

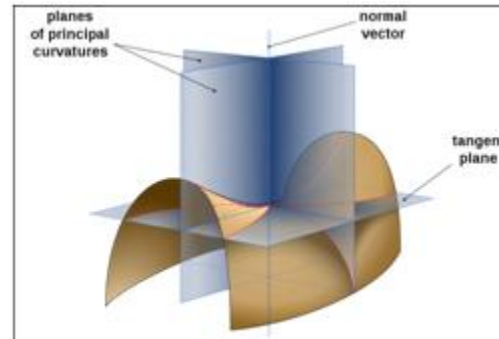


Figure VI

The principal curvatures of a surface at a point are key components in deriving the mean (H) curvature and Gaussian (K) curvature for the surface. (Weisstein, Curvature)

$$K = (k_1 k_2)$$

$$H = \frac{1}{2}(k_1 + k_2)$$

Mean curvature is an extrinsic quality of a curve, whereas Gaussian curvature is intrinsic. Before further exploring these curvatures mathematically, a discussion of intrinsic and extrinsic curvature is in order.

Intrinsic vs. Extrinsic Curvature

As discussed earlier, the properties of curves fall into two main categories: intrinsic and extrinsic. The most colorful explanation of the difference between the two occurs within the context of the story of A. Square, a heretical two-dimensional square inhabiting a plane called “Flatland.” The concept of “Flatland” was first explored in the work, *Flatland: A Romance of Many Dimensions*, by Edwin Abbott Abbott in 1884, but the adventures of A. Square (Figure VII (Rucker, 1977, p. 4)) and his interactions with A. Sphere and A. Polygon made such a lasting impression on the study of differential geometry that many other authors have utilized A. Square, his cohorts, and his reality to illustrate what 2-dimensional inhabitants might observe and conclude about the geometry of their own world.

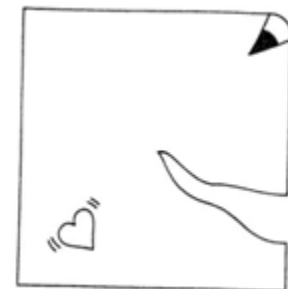


Figure VII

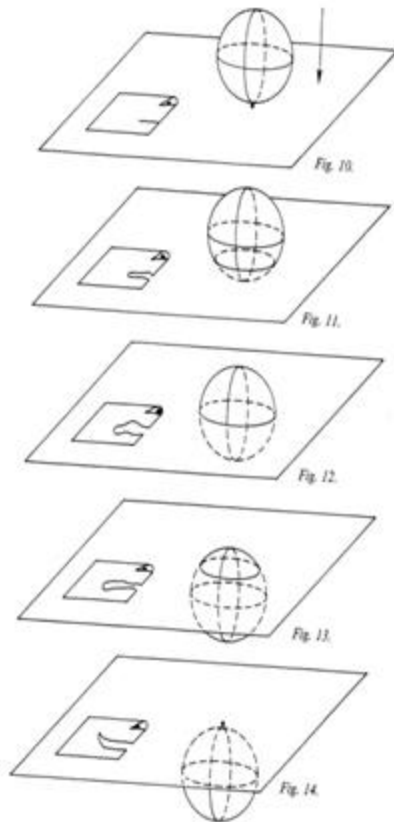


Figure VIII

Within the context of “Flatland,” the intrinsic qualities of the surface of their world could be detected by the two dimensional inhabitants, whereas extrinsic qualities would be detectible only *externally* to someone with a different perspective – e.g. the three dimensional sphere (A. Sphere) who interacts with A. Square in an attempt to explain to him the nature of his own reality. In one instance A. Sphere tries to demonstrate his own shape by passing through the plane of Flatland before A. Square’s eyes. In Figure VIII (Rucker, 1977, p. 5), A. Square observes him first as a dot, and then as a circle of increasing diameter.

Within a mathematical context, extrinsic curvature is dependent on the embedding of the surface in another space (\mathbb{R}^3 or \mathbb{S}^2 for example) whereas intrinsic curvature exists independent of this embedding.

Fourth Dimension

Supposing that A. Hypersphere wanted to mess around with those of us here in the third dimension, he might very well consider robbing a bank. In the same way that it would be simple for A.

Sphere to reach into A. Square’s 2-dimensional locked safe and take his money without A. Square ever being the wiser until he went to make a withdrawal, so too might A. Hypersphere cause quite a bit of consternation by “disappearing” the treasure in Fort Knox (or a nuclear missile or two!)

First Fundamental Form

The three fundamental forms can be used to determine the metric properties of an object. The third fundamental form can be derived from the first and second forms. Surfaces can be described by multiple properties, among them Gaussian curvature, mean curvature, line element, area element, and normal curvature. Each of these is a metric property that the fundamental forms help to define mathematically. Gaussian and mean curvature will be discussed in more detail later on.

If we define the length of a curve, $Z(t) = Z(u(t), v(t))$, on a surface to be (J.J.Stoker, 1956)

$$s = L_{t_0}^{t_1} = \int_{t_0}^{t_1} \sqrt{(Z'(t))^2} dt$$

With $Z'(t) = Z_u \frac{du}{dt} + Z_v \frac{dv}{dt}$ we have

$$\left(\frac{ds}{dt}\right)^2 = Z' \cdot Z' = E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2$$

Where $E = Z_u^2 = Z_u \cdot Z_u$, $F = Z_u \cdot Z_v$, $G = Z_v^2$.

The First Fundamental Form refers to the quadratic on the right of the equation. It is positive definite. (J.J.Stoker, 1956)

Second Fundamental Form

The Second Fundamental Form is another extrinsic property of surface. It depends on an embedding into Euclidean N-space of $N \geq 3$. It is defined as the square of the Euclidean distance from a point close to the one being considered to the tangent plane. **Error! Reference source not found.** Figure IX (J.J.Stoker, 1956) illustrates these values. It measures the deviation of neighboring points on the surface from the tangents plane at a specific point and has the form (J.J.Stoker, 1956):

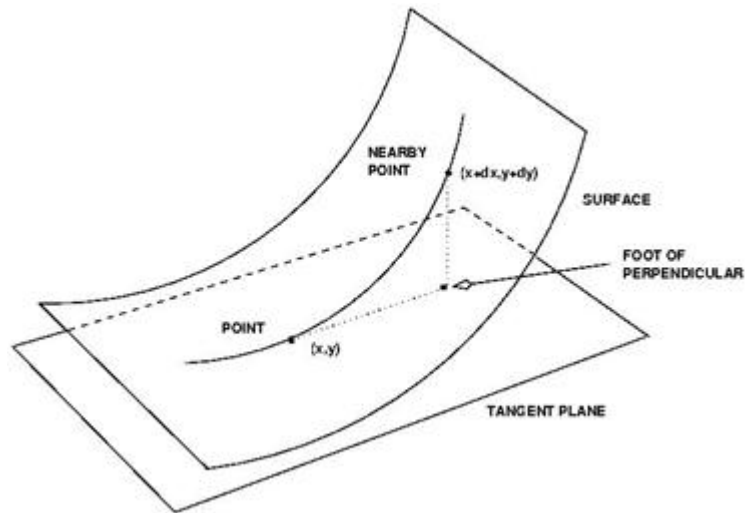


Figure IX

$$II = edu^2 + 2fdxdy + gdy^2$$

Where

$$e = -\bar{N}_u \cdot \bar{Z}_u = \bar{N} \cdot \bar{Z}_{uu}$$

$$f = -\bar{N}_v \cdot \bar{Z}_u = \bar{N} \cdot \bar{Z}_{uv} = \bar{N} \cdot \bar{Z}_{vu} = -\bar{N}_u \cdot \bar{Z}_v$$

$$g = -\bar{N}_v \cdot \bar{Z}_v = \bar{N} \cdot \bar{Z}_{vv}$$

Mean Curvature

As stated earlier, mean curvature is an extrinsic property of a surface derived from the principal curvatures of the surface. Mean curvature can also be stated in terms of the coefficients of the first and second fundamental forms (Weisstein, Mean Curvature):

$$H = \frac{eG - 2F + gE}{2(EG - F^2)}$$

Gaussian Curvature

As remarked earlier, Gaussian Curvature was truly innovative because of Carl Friedrich Gauss's discovery that that it could be understood intrinsically to the surface, which he stated in his *Theorem Egregium*.

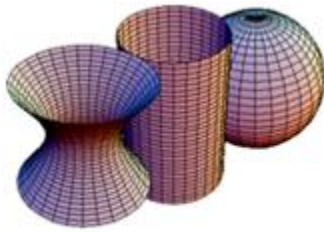


Figure X

One informal example of Gaussian Curvature in action would be if the inhabitant of a surface traced out a circle, and found the circumference of that circle to be less than $2\pi r$. The inhabitants could draw the conclusion from this that their surface was positively curved – due to this measurement's deviation from what would be expected on a flat surface. The sign of Gaussian Curvature at a point informs the nature of the surface at that point. In Figure X (Jhausauer, 2007) the Gaussian curvatures of the shapes from left to right are negative, zero, and positive. On a regular

patch, Gaussian curvature in terms of the coefficients of the first and second fundamental forms can be defined as (Weisstein, Gaussian Curvature):

$$K = \frac{eg - f^2}{EG - F^2}$$

A Monge Patch Application

A Monge patch is nothing more than a local surface with very specific properties.

When examining a surface in a Monge patch, the calculations of the mean and Gaussian curvature are in a more accessible form. A Monge patch is a local surface where $x: U \rightarrow \mathbb{R}^3$ of the form $x(u, v) = (u, v, h(u, v))$, where U is an open set in \mathbb{R}^2 and h is a function that is differentiable in \mathbb{R} . By applying the Monge patch to the first fundamental form, the coefficients are now given by (Weisstein, Monge Patch):

$$E = 1 + h_u^2$$

$$F = h_u h_v$$

$$G = 1 + h_v^2$$

Similarly, applying the Monge patch to the second fundamental form, the coefficients become the following (Weisstein, Monge Patch):

$$e = \frac{h_{uu}}{\sqrt{1 + h_u^2 + h_v^2}}$$

$$f = \frac{h_{uv}}{\sqrt{1 + h_u^2 + h_v^2}}$$

$$g = \frac{h_{vv}}{\sqrt{1 + h_u^2 + h_v^2}}$$

Now the mean (H) curvature and Gaussian (K) curvature for a Monge patch can be defined to be

$$H = \frac{(1 + h_v^2)h_{uu} - 2h_u h_v h_{uv} + (1 + h_u^2)h_{vv}}{(1 + h_u^2 + h_v^2)^{\frac{3}{2}}}$$

$$K = \frac{h_{uu}h_{vv} - h_{uv}^2}{(1 + h_u^2 + h_v^2)^2}$$

Total Curvature

According to Gauss' *Theorem Egregium*, the total curvature K at any point P on a surface depends only on the values of E, F, and G at P and their derivatives from the first and second fundamental forms. (University of Waterloo, 1996)

Original Examples of Curvature

A Function f of x and y

i. Let $f(x, y) = \cos(x) + \sin(y)$

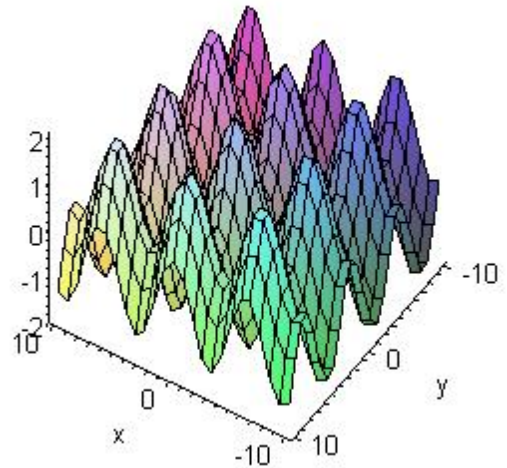
$u = x$ $h_u = -\sin(u)$

$v = y$ $h_v = \cos(v)$

$h = \cos(u) + \sin(v)$ $h_{uu} = -\cos(u)$

$h_{vv} = -\sin(v)$

$h_{uv} = 0$



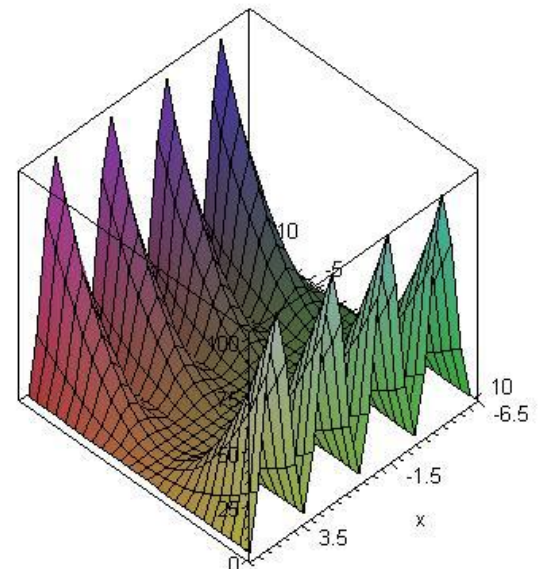
Therefore,

$$H = \frac{(1 + \cos^2(v)) - \cos(u) - 2(-\sin(u) \cos(v) (0)) + (1 + \sin^2(u)) - \sin(v)}{(1 + \sin^2(u) + \cos^2(v))^{\frac{3}{2}}}$$

$$= \frac{-\cos(u) - \cos(u) \cos^2(v) - \sin(v) + \sin(v) \sin^2(u)}{(1 + \sin^2(u) + \cos^2(v))^{\frac{3}{2}}}$$

$$K = \frac{-\cos(u) - \sin(v) - 0}{(1 + \sin^2(u) + \cos^2(v))^2}$$

$$= \frac{\cos(u) \sin(v)}{(1 - \sin(u) + \cos(v))^2}$$



II. Let $f(x, y) = y^2 \sin^2(x)$

$$u = x \quad h_u = 2y^2 \sin(x) \cos(x) = y^2 \sin(2x)$$

$$v = y \quad h_v = 2y \sin^2(x)$$

$$h = y^2 \sin^2(x) \quad h_{uu} = -2y^2 \sin^2(x) + 2y^2 \cos^2(x)$$

$$h_{vv} = 2 \sin^2(x)$$

$$h_{uv} = 4y \cos(x) \sin(x) = 2y \sin(2x)$$

Therefore,

$$H = \frac{(1 + 4y^2 \sin^4(x))(-2y^2 \sin^2(x) + 2y^2 \cos^2(x)) - (32y^4 \cos^2(x) \sin^4(x)) + (1 + 4y^4 \sin^2(x) \cos^2(x))2 \sin^2(x)}{(1 + 4y^4 \sin^2(x) \cos^2(x) + 4y^2 \sin^4(x))^{\frac{3}{2}}}$$

$$H = \frac{2y^2 \cos^2(x) - 2y^2 \sin^2(x) + 8y^4 \sin^4(x) \cos^2(x) - 8y^4 \sin^6(x) - 32y^4 \cos^2(x) \sin^4(x) + 2 \sin^2(x) + 8y^4 \sin^4(x)}{(1 + 4y^4 \sin^2(x) \cos^2(x) + 4y^2 \sin^4(x))^{\frac{3}{2}}}$$

$$H = \frac{2y^2 \cos^2(x) - 2y^2 \sin^2(x) - 8y^4 \sin^6(x) - 16y^4 \sin^2(2x) \sin^2(x) + 2 \sin^2(x)}{(1 + 4y^4 \sin^2(x) \cos^2(x) + 4y^2 \sin^4(x))^{\frac{3}{2}}}$$

$$K = \frac{h_{uu} h_{vv} - h_{uv}^2}{(1 + h_u + h_v)^2}$$

$$K = \frac{2y^2 \sin^2(x) + 2y^2 \cos^2(x) 2 \sin^2(x) - 4y^2 \sin^2(2x)}{(1 + y^2 \sin(2x) + 2y \sin^2(x))^2}$$

$$K = \frac{2y^2 \sin^2(x) - 3y^2 \sin^2(2x)}{(1 + y^2 \sin(2x) + 2y \sin^2(x))^2}$$

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